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Betti numbers of compressed level algebras

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Abstract

We present a conjecture on the generic Betti numbers of level algebras. We prove the conjecture in the case of Gorenstein Artin algebras of embedding dimension four, and in the case of Artin level algebras whose socle dimension is large. Furthermore, we present computational evidence for the conjecture. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Level algebras – introduced by Stanley [20] – are often encountered in various applications of commutative algebra, such as algebraic geometry and algebraic combinatorics. The level algebras of maximal Hilbert function among all level algebras of given codimension and socle type are called *compressed* level algebras and fill a non-empty Zariski open set in the natural parameter space.

Here we will study the Betti numbers of compressed level algebras. Though the Hilbert function is known it is still very hard to determine the Betti numbers completely. We give lower bounds for these Betti numbers and we conjecture that these bounds are sharp.

For the coordinate ring of a generic set of points in projective space there is a similar conjecture, the Minimal Resolution Conjecture, MRC, by Lorenzini [18]. Much work has been done on the MRC including a recent result by Hirschowitz and Simpson [14] stating that it holds if the number of points is large enough compared to the embedding dimension.

The Minimal Resolution Conjecture implies part of our conjecture but there are also some cases where it seems as the MRC does not hold while our conjecture still holds.

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In Section 3.4 we will show the relation between the case of compressed algebras and the case of generic points.

We are able to prove our conjecture in two cases.

- When the socle is large we can apply the theorem on linear syzygies of generic forms by Hochster and Laksov [15] to prove the conjecture.
- For Gorenstein Artin algebras of codimension 4, we use the Minimal Resolution Conjecture for points in P^3 proved by Ballico and Geramita [1], and the embedding of the canonical module of the coordinate of points as an ideal of a certain initial degree to obtain the conjecture.

We also present extensive computational evidence for our conjecture in terms of calculations done with the computer algebra system *Macaulay* [2].

1.1. Compressed level algebras

In this section we recall some notations and basic results on compressed level algebras. References are Iarrobino [16, 17], Fröberg and Laksov [10], and a joint work with Laksov [5].

Setup 1.1. Let $R=k[x_1,x_2,\dots,x_r]$ be the polynomial ring in r independent variables $x_1,x_2,\dots,x_r$ over an infinite field k. We give R the natural grading by assigning all variables degree 1. Denote by $\mathscr{M}_d$ the monomials of degree d in R. For any k-subspace V in R_c, we write

$$(V:R)_d=\{a\in R_d \mid ab\in V, \text{ for all } b\in R_{c-d}\}, \tag{1.1}$$

for $d=0,1,\dots$ Let I be a homogeneous ideal of R and let $A=R/I$. The *socle* of A is defined by $\operatorname{Soc} A=\{a\in A \mid ab=0, \text{ for all } b \text{ in } R_1\}$, where we look at A as an R-module. We say that an Artin algebra $A=R/I$ is *level* if its socle is concentrated in one degree c, which is the highest non-empty degree of A. Furthermore, A is *Gorenstein* if $\dim_k \operatorname{Soc} A=1$.

Recall the following fact on level algebras (cf. [5, Proposition 2.4]).

Proposition 1.2. *Let $A=R/I$ be a graded Artin quotient of R. Then A is level with socle in degree c if and only if there is a subspace $V\subseteq R_c$ such that $I=\bigoplus_{d\geq 0}(V:R)_d$.*

We now recall the definition of compressed level algebra (cf. [16, 10]).

Definition 1.3. Let A be an Artin level algebra quotient of R with socle in degree c. Then A is *compressed* if its Hilbert function is given by

$$H_A(d)=\dim_k A_d=\min\{\dim_k R_d, s\ \dim_k R_{c-d}\}, \tag{1.2}$$

for $d=0,1,\dots,c$, where s is the k-dimension of the socle.

The Hilbert function of a compressed algebra A is the maximal possible among all algebra whose socle is of the same type as the socle of A. In fact there are compressed level algebras, according to the following proposition (cf. [16, 5, Theorem 3.4]).

Proposition 1.4. *Let V be a codimension s subspace of R_c in general position, and let $I = \bigoplus_{d \geq 0}(V : R)_d$. Then $A = R/I$ is a compressed Artin level algebra with socle of dimension s in degree c.*

2. Minimal free resolutions and Betti numbers

In this section we give the notations and basic results needed for working with level algebras, minimal free resolutions and Betti numbers. A suitable reference for most of this material is *Cohen–Macaulay Rings* by Bruns and Herzog [6].

Setup 2.1. Let $R = k[x_1, x_2, \dots, x_r]$ be the polynomial ring in r independent variables $x_1, x_2, \dots, x_r$ over k. Denote by ω_R the canonical module $R(-r)$ of R. If M is an artinian finitely generated R-module, we define the dual module $M^\vee$ as $\mathrm{Hom}_k(M, k)$. The module $M^\vee$ has an R-module structure defined by $x\phi(y) = \phi(xy)$ for all $x, y \in R$ and all $\phi \in M^\vee$. The grading of $M^\vee$ is naturally given by $M^\vee_d = \mathrm{Hom}_k(M_{-d}, k)$, for all $d \in Z$. Thus if M is graded by non-negative integers, $M^\vee$ is graded by non-positive integers. In the same way we can define a graded R-module $\mathrm{Hom}_R(M, R)$ for all graded R-modules M. We observe that k is a graded R-module via the augmentation map $R \to k$, and thus it is zero except in degree 0.

Let M be a finitely generated graded R-module. By the Hilbert Syzygy Theorem [6, Corollary 2.2.14] there is a finite free graded resolution

$$0 \to \bigoplus_{j=1}^{b_r} R(-n_{j,r}) \xrightarrow{\partial_r} \cdots \xrightarrow{\partial_2} \bigoplus_{j=1}^{b_1} R(-n_{j,1}) \xrightarrow{\partial_1} \bigoplus_{j=1}^{b_0} R(-n_{j,0}) \xrightarrow{\partial_0} M \to 0, \tag{2.1}$$

where $b_0, b_1, \dots, b_r$ are the *Betti numbers* of M and $\{n_{i,j}\}$ are the *numerical characters* of M, under the assumption that the resolution is *minimal*, i.e., if all the entries of the matrices of ∂_i are in the irrelevant maximal ideal $\mathfrak{m}$ of R. The numerical characters are uniquely defined by M if we assume that $n_{j+1,i} \geq n_{j,i}$ (cf. [6, Proposition 1.5.16]).

In the following proposition we have collected some facts that will be very useful in the study of Betti numbers of Artin level algebras. (cf. [6, Theorems 3.6.17 and 3.6.19] for (i)–(v) and Fröberg and Laksov [10] for (vi)–(viii).)

Proposition 2.2. *Let M be an artinian finitely generated graded R-module. Then the following hold:*

(i) *$M^\vee$ is artinian and finitely generated.*
(ii) $\mathrm{Ext}^i_R(M, \omega_R) = 0$ *for* $i < r$.
(iii) $M^\vee \cong \mathrm{Ext}^r_R(M, \omega_R)$.

(iv) *If $F_\bullet$ is a minimal free resolution of M, then $F_\bullet^*$ is a minimal free resolution of $M^\vee$.*

(v) $\mathrm{Tor}_i^R(M,k)_d \cong \mathrm{Tor}_{r-i}^R(M^\vee,k)_{r-d}$, *for* $i=0,1,\dots,r$ *and* $d\in Z$.

(vi) $n_{1,i}<n_{1,i+1}$ *and* $n_{b_i,i}<n_{b_{i+1},i+1}$, *for* $i=0,1,\dots,r-1$.

(vii) *The numerical characters of M determine the Hilbert function of M by*

$$\mathrm{Hilb}_M(z)=\sum_{i=0}^{r}\sum_{j=1}^{b_i}\frac{(-1)^i z^{n_{j,i}}}{(1-z)^r}. \tag{2.2}$$

(viii) *If the numerical characters of M satisfy $n_{b_i,i}<n_{1,i+1}$, for $0\le i<r$. Then they are determined by the Hilbert function of M.*

We recall some notation for the Koszul complex, which is a minimal free resolution of k over R (cf. [6, Section 1.6]).

Definition 2.3. Let W be a vector space of dimension r over k with a basis given by $\{e_1,e_2,\dots,e_r\}$. Then we can define the Koszul complex, $K_\bullet$, of $x_1,x_2,\dots,x_r$ as follows: For any $i=1,2,\dots,r$, define $K_i=\bigwedge^i W\otimes R(-i)$ and define the R-linear map δ_i: $\bigwedge^i W\otimes R(-i)\to\bigwedge^{i-1}W\otimes R(-i+1)$ by

$$\delta_i(e_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_i})=\sum_{l=1}^{i}(-1)^{l+1}e_{j_1}\wedge\cdots\wedge\hat{e}_{j_l}\wedge\cdots\wedge e_{j_i}\otimes x_{j_l}. \tag{2.3}$$

We will use the notation $\varepsilon_{j_1,j_2,\dots,j_i}=e_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_i}$ for the natural basis elements in $\bigwedge^i W$. Furthermore, we will write $\mathscr{S}(i)$ for the set of ordered i-tuples $(j_1,j_2,\dots,j_i)$, with $1\le j_1<j_2<\cdots<j_i\le r$.

Since the Koszul complex, $K_\bullet$, is a minimal free resolution of k as an R-module it gives us the possibility to compute $\mathrm{Tor}_i^R(A,k)$ as $H_i(A\otimes_R K_\bullet)$.

We will need the following two propositions, proved by Cavaliere et al. [8, Proposition 1; 9, Proposition 1.3].

Proposition 2.4. *Let $e_1,e_2,\dots,e_r$ be a basis for the vector space W and let $K_\bullet$ be the Koszul complex defined in* (2.3). *Then we have that a basis for the image of $\delta_i:\bigwedge^i W\otimes R_{t-1}\to\bigwedge^{i-1}W\otimes R_t$ is given by the images of $e_{j_1}\wedge\cdots\wedge e_{j_i}\otimes m$ where $j_1<j_2<\cdots<j_i$ and m is a monomial of degree t in the variables $x_{j_1},x_{j_1+1},\dots,x_r$. The dimension of* $\mathrm{im}(\delta_i)$ *is* $\binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i}$.

Proposition 2.5. *Let I be a homogeneous ideal in R of initial degree t and let $A=R/I$. Then we have*

$$\mathrm{Tor}_i^R(A,k)_{i+t-1}\cong\ker(\pi)\cap\ker(\delta_{i-1})\quad \textit{for } i=2,3,\dots,r, \tag{2.4}$$

where δ_{i-1} is the map $\bigwedge^{i-1} W \otimes R_t \to \bigwedge^{i-2} W \otimes R_{t+1}$ of the Koszul complex and π is the quotient map $\bigwedge^{i-1} W \otimes R_t \to \bigwedge^{i-1} W \otimes A_t$.

We also recall that there is a homological criterion on an Artin algebra to be level (cf. [10]).

Proposition 2.6. *Let $A = R/I$ be a graded Artin algebra. Then there is an isomorphism of graded R-modules $\operatorname{Soc} A(-r) \xrightarrow{\sim} \operatorname{Tor}_r^R(A,k)$. In particular, A is level if and only if there is an integer c such that $\operatorname{Tor}_r^R(A,k)_d = 0$, for $d \neq r + c$.*

Remark 2.7. Proposition 2.6 provides us with an alternative definition of level algebras, which is useful even in the higher dimensional Cohen–Macaulay case. We can see this from the isomorphisms between the Tor_i^R-modules of A and the $\operatorname{Tor}_i^{R'}$-modules of the artinian reduction $A' = A/(a_1, a_2, \dots, a_n)$ of A, where $R' = R/(y_1, y_2, \dots, y_n)$, and $y_j \mapsto a_j$, for $j = 1, 2, \dots, n$. In fact, we have that if $F_\bullet$ is a free resolution of A, then $F_\bullet \otimes_R R'$ is a free resolution of A' and the result follows since $F_\bullet \otimes_R k \cong F_\bullet \otimes_R (R' \otimes_{R'} k) \cong (F_\bullet \otimes_R R') \otimes_{R'} k$.

3. Betti numbers of compressed Artin level algebras

In the following sections we will investigate the Betti numbers of compressed Artin level algebras. Because Proposition 1.4 tells us that the generic Artin level algebra is compressed, which means that the Hilbert function is known, it is tempting to suggest that the situation for the Betti numbers is similar. This is true in the sense that all generic compressed Artin level algebras, of given embedding dimension and socle type, have the same Betti numbers. The difference from the case of the Hilbert functions of compressed algebras is that is not so easy to determine these Betti numbers.

We start by examining a very special case of compressed Artin level algebra which can be expressed as $R/\mathfrak{m}^{c+1}$ where $\mathfrak{m}$ is the irrelevant maximal ideal of R. Thus we have that the initial degree of the kernel $R \to A$ is $c + 1$. The reason why we treat this case separately is that it will be convenient to assume that the initial degree of compressed algebras is at most c, where c is the degree of the socle.

Proposition 3.1. *Let $A = R/\mathfrak{m}^{c+1}$. Then $\operatorname{Tor}_i^R(A,k)$ is concentrated in degree $i+c$, for $i = 1, 2, \dots, r$ and the Betti numbers of A are given by*

$$b_i = \dim_k \operatorname{Tor}_i^R(A,k)_{i+c} = \binom{c+i-1}{i-1}\binom{c+r}{r-i} \quad \text{for } i = 1, 2, \dots, r. \tag{3.1}$$

Proof. Since the socle of A is contained in degree c we have from Proposition 2.6 that $\operatorname{Tor}_r^R(A,k)$ is concentrated in degree $c + r$. The generators of the ideal $\mathfrak{m}$ all have degree c and therefore $\operatorname{Tor}_1^R(A,k)$ is concentrated in degree $c+1$. Now it follows from Proposition 2.2(vi) that $\operatorname{Tor}_i^R(A,k)$ is concentrated in degree $i+c$ for $i=1,2,\ldots,r$.

We now use Proposition 2.5 to compute the Betti numbers of A. The quotient map $\pi:\bigwedge^{i-1} W\otimes R_{c+1}\to\bigwedge^{i-1} W\otimes A_{c+1}$ is the zero map since the kernel is all of R_{c+1}. Hence we have that $\operatorname{Tor}_i^R(A,k)\cong\ker(\delta_{i-1})=\operatorname{im}(\delta_i)$, whose dimension is given by Proposition 2.4. □

We can now give a homological criterion on a level algebra to be compressed (cf. [10, Proposition 16] for one direction).

Proposition 3.2. *Let $A=R/I$ be an Artin level algebra with socle in degree c and let t be the initial degree of I. Then A is compressed if and only if $\operatorname{Tor}_i^R(A,k)$ is concentrated in degrees $t+i-1$ and $t+i$, for $i=1,2,\ldots,r-1$.*

Proof. We note that $\operatorname{Tor}_1(A,k)=0$ in degrees less than t is equivalent to say that $\dim_k A_d=\dim_k R_d$ for $d<t$. Let $s=\dim_k \operatorname{Soc} A=\dim_k A_c$. We now look at the dual module $A^\vee(-c)$. We have that this module is a quotient of R^s, which means that $\operatorname{Tor}_1^R(A^\vee(-c),k)=0$ in degrees less than $c+1-t$ if and only if $\dim_k A^\vee(-c)_d=\dim_k R^s_d=s\dim_k R_d$, for $d<c+1-t$. Using Proposition 2.2(v) we see that

$$\operatorname{Tor}_1^R(A^\vee(-c),k)_d=\operatorname{Tor}_{r-1}^R(A,k)_{r+c-d}. \tag{3.2}$$

Hence $\operatorname{Tor}_{r-1}^R(A,k)=0$ in degrees greater than $r-1+t$ is equivalent to the condition that $\operatorname{Tor}_1^R(A^\vee(-c),k)=0$ in degrees less than $c+1-t$. But this is equivalent to say that $\dim_k A^\vee(-c)_d=s\dim_k R_d$, for $d<c+1-t$. Moreover, $\dim_k A^\vee(-c)_d=\dim_k A_{c-d}$. Hence we have that $\operatorname{Tor}_{r-1}^R(A,k)=0$ in degrees greater than $r-1+t$ is equivalent to $\dim_k A_d=s\dim_k R_{c-d}$, for $d>t-1$.

By Proposition 2.2(vi) we have that $\operatorname{Tor}_i^R(A,k)$ is concentrated in degrees $i+t-1$ and $i+t$, for $i=1,2,\ldots,r-1$ if and only if $\operatorname{Tor}_1^R(A,k)_d=0$, for $d<t$ and $\operatorname{Tor}_{r-1}^R(A,k)_d=0$, for $d>r-1+t$.

By the argument above, the latter conditions are equivalent to say that $\dim_k A_d=\dim_k R_d$, for $d=0,1,\ldots,t-1$ and that $\dim_k A_d=s\dim_k R_{c-d}$, for $d=t,t+1,\ldots,c$. Hence the proposition follows. □

Because of the special form of the Hilbert series of compressed level algebras we are able to use Eq. (2.2) to get the following proposition. There is a similar result for *short* graded algebras due to Cavaliere et al. [9, Proposition 1.6].

Proposition 3.3. *Let $A=R/I$ be a compressed graded Artin level algebra of embedding dimension r with socle in degree c of dimension s such that the initial degree* *of I is t. Then we have that*

$$\begin{aligned}&\dim_k \operatorname{Tor}_i^R(A,k)_{i+t-1} - \dim_k \operatorname{Tor}_{i-1}^R(A,k)_{i+t-1}\\ &= \binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i} - s\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1},\\ &\textit{for } i=1,2,\ldots,r. \qquad (3.3)\end{aligned}$$

Proof. Because of Proposition 3.1 we can assume that $t \le c$. We will use the identity

$$(1-z)^r \sum_{d=0}^{l} \binom{r-1+d}{r-1} z^d = 1 + \sum_{j=1}^{r} (-1)^j \binom{l+j-1}{j-1}\binom{l+r}{r-j} z^{l+j}. \qquad (3.4)$$

By Proposition 2.2(vii) we have that

$$(1-z)^r \operatorname{Hilb}_A(z) = \sum_{i=0}^{r}\sum_{j=1}^{b_i} (-1)^i z^{n_{j,i}} = \sum_{i=0}^{r}\sum_{d=0}^{\infty} (-1)^i \dim_k \operatorname{Tor}_i^R(A,k)_d z^d. \qquad (3.5)$$

Since A is compressed we have $\dim_k A_d = \min\{\binom{r-1+d}{r-1}, s\binom{r-1+c-d}{r-1}\}$, which yields that

$$(1-z)^r \operatorname{Hilb}_A(z) = (1-z)^r \sum_{d=0}^{t-1} \binom{r-1+d}{r-1} z^d + s(1-z)^r \sum_{d=0}^{c-t} \binom{r-1+d}{r-1} z^{c-d}. \qquad (3.6)$$

Since $t \le c$ it follows from (3.4) and (3.6) that

$$\begin{aligned}&(1-z)^r \operatorname{Hilb}_A(z)\\ &= 1 + \sum_{i=1}^{r} (-1)^i \binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i} z^{t-1+i}\\ &\quad + s(-1)^r z^{r+c} + s(-1)^r z^{r+c} \sum_{i=1}^{r} (-1)^i \binom{c-t+i-1}{i-1}\binom{c-t+r}{r-i} z^{t-c-i}\\ &= 1 + s(-1)^r z^{r+c}\\ &\quad + \sum_{i=1}^{r} (-1)^i \left(\binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i}\right.\\ &\qquad \left. - s\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1}\right) z^{i+t-1}. \qquad (3.7)\end{aligned}$$

From Proposition 3.2 we have that, for $i = 1,2,\dots,r-1$, the last sum in (3.5) is taken from $d = t+i-1$ to $d = t+i$. Hence it becomes

$$1 + \sum_{i=1}^{r-1} (-1)^i \dim_k \operatorname{Tor}_i^R(A,k)_{i+t-1} z^{i+t-1}$$
$$+ \sum_{i=1}^{r-1} (-1)^i \dim_k \operatorname{Tor}_i^R(A,k)_{i+t} z^{i+t} + s(-1)^{r+c} z^{r+c}. \tag{3.8}$$

The proposition now follows from equating the coefficients of z^{i+t-1} in Eq. (3.8) with the same coefficient in the final expression of (3.7). □

Because of Proposition 3.2 it is convenient to introduce the following notation.

Notation 3.4. Let A be a compressed Artin level algebra of embedding dimension r. Then we define $b_i' = \dim_k \operatorname{Tor}_i^R(A,k)_{i+t-1}$ and $b_i'' = \dim_k \operatorname{Tor}_i^R(A,k)_{i+t}$, for $i = 1,2,\dots, r-1$. We sometimes express the Betti numbers of a compressed Artin level algebra as

$$\begin{pmatrix} b_1' & b_2' & \dots & b_{r-1}' \\ b_1'' & b_2'' & \dots & b_{r-1}'' \end{pmatrix}. \tag{3.9}$$

Definition 3.5. Let $A = R/I$ be a compressed graded Artin level algebra of codimension r with socle of dimension s in degree c. Then A is *extremely compressed* (cf. [10]) if there is an integer d_0 such that

$$\binom{r-1+d_0}{r-1} = s\binom{r-1+c-d_0}{r-1} \tag{3.10}$$

Observe that, since A is compressed, the initial degree of I is $t = d_0 + 1$.

The following proposition is a special case of a result by Buchsbaum et al. [16, Proposition 4.1], where they prove that under certain numerical conditions on the socle type of a compressed algebra the resolution becomes *almost linear* which means that all the maps in the resolution is of degree 1 except for the first and the last. See also [10, Proposition 16].

Proposition 3.6. *Let $A = R/I$ be an extremely compressed Artin level algebra. Then we have that $n_{j,i} = t+i-1$, for all i,j, where t is the initial degree of I.*

Furthermore, the Betti numbers of A are

$$b_i = \binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i} - s\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1}, \tag{3.11}$$

for $i = 1,2,\dots,r-1$.

3.1. Existence of generic Betti numbers for level algebras

As we saw in Proposition 3.6 the Betti numbers of an extremely compressed level algebra are uniquely determined by the Hilbert function, or by the embedding dimension together with the degree and dimension of the socle. We could hope that this is the case also for compressed level algebras in general, but this is not true. There are many possibilities for the Betti numbers of compressed level algebras with the same Hilbert function.

However, inspired by Proposition 1.4, we will prove that there are generic Betti numbers of compressed level algebras, in the sense that these Betti numbers are obtained by $A=R/\bigoplus_{d\geq 0}(V:R)_d$, for a generic subspace $V\subseteq R_c$. More precisely, we will prove that there is a non-empty open set in the Grassmannian parametrizing all codimension s subspaces of R_c such that all algebras $A=R/\bigoplus_{d\geq 0}(V:R)_d$ with V in this open set have the same Betti numbers. This is not very surprising and the hard problem is of course to actually find these generic Betti numbers. The situation is similar to the case of the coordinate ring of a generic set of points in projective space as we will see in Section 3.4.

The way we will prove the existence of generic Betti numbers is to produce a matrix with generic coordinates such that the rank of this matrix determines a certain Betti number (cf. Cavaliere et al. [9, Proposition 2.1].)

Definition 3.7. Given integers s and c. Let $\eta_m^{(j)}$, $j=1,2,\dots,s$, $m\in\mathcal{M}_c$ be independent variables over k. Let t be the least positive integer such that $s\binom{c-t+r-1}{r-1}<\binom{t+r-1}{r-1}$. We define $s\binom{r-1+c-t}{r-1}\times\binom{r-1+t-1}{r-1}$-matrices $\boldsymbol{A}_1,\dots,\boldsymbol{A}_r$, as block matrices built by s blocks $\boldsymbol{A}_j^{(l)}$, for $l=1,2,\dots,s$, where $(\boldsymbol{A}_j^{(l)})_{m',m}=\eta^{(l)}_{x_jm'm}$, for all $m'\in\mathcal{M}_{c-t}$ and all $m\in\mathcal{M}_{t-1}$.

Furthermore, for $i=2,3,\dots,r-1$, we define a block matrix $\boldsymbol{M}_i$ with $\binom{r}{i-1}\times\binom{r}{i}$ blocks $\boldsymbol{A}_{\alpha,\beta}$ where $\alpha\in\mathcal{S}(i-1)$ and $\beta\in\mathcal{S}(i)$ (recall $\mathcal{S}(i)$ from Definition 2.3). These blocks are defined by

$$\boldsymbol{A}_{\alpha,\beta}=\begin{cases}(-1)^{l+1}\boldsymbol{A}_{j_l} & \text{if } \beta=(j_1,j_2,\dots,j_i) \text{ and } \alpha=(j_1,\dots,\hat{j}_l,\dots,j_i),\\ 0 & \text{if } \alpha\not\subset\beta.\end{cases} \tag{3.12}$$

We observe that the entries of $\boldsymbol{M}_i$ are given by

$$(\boldsymbol{M}_i)_{(\alpha,j,m'),(\beta,m)}=\begin{cases}(-1)^{l+1}\eta^{(j)}_{x_{j_l}m'm}, & \text{if } \alpha=(j_1,\dots,\hat{j}_l,\dots,j_i),\\ 0 & \text{if } \alpha\not\subset\beta,\end{cases} \tag{3.13}$$

where $\beta=(j_1,j_2,\dots,j_i)$.

We sometimes want to specialize the variables $\eta_m^{(j)}$ to elements $\lambda_m^{(j)}$ in k. We will then denote the resulting matrix by $\boldsymbol{M}_i^{\lambda}$.

Recall from Definition 2.3 that for $\alpha=(j_1,j_2,\dots,j_i)\in\mathcal{S}(i)$ we denote $e_{j_1}\wedge\cdots\wedge e_{j_i}$ by ε_α.

Proposition 3.8. *Let $A = R/I$ be an Artin level algebra with socle of dimension s in degree c and let $V = I_c$ be given by*

$$V = \left\{ \sum_{m \in \mathcal{M}_c} \xi_m m \;\middle|\; \sum_{m \in \mathcal{M}_c} \lambda_m^{(j)} \xi_m = 0, \text{ for } j = 1, 2, \ldots, s \right\}, \tag{3.14}$$

for some elements $\lambda_m^{(j)}$, $j = 1, 2, \ldots, s$, $m \in \mathcal{M}_c$ of k. Then we have that $\ker(\pi) \cap \operatorname{im}(\delta_i)$ *from Proposition 2.5 is given by all elements of the form*

$$\sum_{\substack{m \in \mathcal{M}_{t-1} \\ \beta \in \mathcal{S}(i)}} \xi_m^{(\beta)} \delta_i(\varepsilon_\beta) \tag{3.15}$$

where $\xi_m^{(\beta)} \in k$, for $m \in \mathcal{M}_{t-1}$ and $\beta \in \mathcal{S}(i)$, satisfying

$$\sum_{\substack{m \in \mathcal{M}_{t-1} \\ \beta \in \mathcal{S}(i)}} (\boldsymbol{M}_i^\lambda)_{(\alpha, j, m'), (\beta, m)} \xi_m^{(\beta)} = 0, \tag{3.16}$$

for $\alpha \in \mathcal{S}(i-1)$, $1 \le j \le s$ and $m' \in \mathcal{M}_{c-t}$.

Proof. The image of the map $\delta_i : \bigwedge^i W \otimes R_{t-1} \to \bigwedge^{i-1} W \otimes R_t$ is generated by the set $\{m\delta_i(\varepsilon_\beta) \mid m \in \mathcal{M}_{t-1},\ \beta \in \mathcal{S}(i)\}$. We now look at the map $\pi : \bigwedge^{i-1} W \otimes R_t \to \bigwedge^{i-1} W \otimes A_t$. By Proposition 1.2, the kernel of the map $R_t \to A_t$ is given by $(V : R)_t$, which in coordinates is the set of elements $\sum_{m \in \mathcal{M}_t} \xi_m m$, such that

$$\sum_{m \in \mathcal{M}_t} \lambda_{m'm}^{(j)} \xi_m = 0 \quad \text{for all } j = 1, 2, \ldots, s \text{ and all } m' \in \mathcal{M}_{c-t}. \tag{3.17}$$

Hence we have that $\ker(\pi) \cap \operatorname{im}(\delta_i)$ is given by the set of elements

$$\sum_{\substack{m \in \mathcal{M}_{t-1} \\ \beta \in \mathcal{S}(i)}} \xi_m^{(\beta)} m \delta_i(\varepsilon_\beta), \tag{3.18}$$

where $\xi_m^{(\beta)} \in k$, for all $\beta \in \mathcal{S}(i)$ and all $m \in \mathcal{M}_{t-1}$, such that

$$\sum_{\substack{m \in \mathcal{M}_{t-1} \\ \beta \in \mathcal{S}(i)}} \sum_{l=1}^{i} (-1)^{l+1} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_l} \wedge \cdots \wedge e_{j_i} \xi_m^{(\beta)} \lambda_{x_{j_l} m' m}^{(j)} = 0, \tag{3.19}$$

for all $j = 1, 2, \ldots, s$, and all $m' \in \mathcal{M}_{c-t}$. Eq. (3.16) is a reformulation of (3.19) using the matrix $\boldsymbol{M}_i^\lambda$. $\square$

Proposition 3.9. *Let $A = R/I$ be an Artin level algebra with socle of dimension s in degree c determined by the elements $\lambda_m^{(j)} \in k$, for $j = 1, 2, \ldots, s$ and $m \in \mathcal{M}_c$ as in*

Proposition 3.8. *Then we have that*

$$b_i' = \binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i} - \operatorname{rank} \boldsymbol{M}_i^\lambda, \tag{3.20}$$

$$b_{i-1}'' = s\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1} - \operatorname{rank} \boldsymbol{M}_i^\lambda \tag{3.21}$$

for $i=2,3,\dots,r-1$.

Proof. The Betti numbers b_i' are given by the dimension of $\ker(\pi)\cap \operatorname{im}(\delta_i)$ by Proposition 2.5. Since $\dim_k \operatorname{im}(\delta_i) = \binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i}$, by Proposition 2.4, and since $\ker(\pi)\cap\operatorname{im}(\delta_i)$ is given by (3.16), the first assertion follows from the dimension theorem for vector spaces. The second assertion is a consequence of the first assertion together with Proposition 3.3. □

Corollary 3.10. *There is a non-empty open set U in the Grassmannian parametrizing all codimension s subspaces of R_c, such that the Betti numbers of the level algebra $A = R/\bigoplus_{d=0}^{\infty}(V:R)_d$ are the same for all V in U.*

Proof. We can restrict our attention to compressed algebras, since these are parametrized by a non-empty open set (cf. Proposition 1.4) of the Grassmannian. Hence we have, by Proposition 3.3, that the Betti numbers b_i are determined by the numbers b_i' and b_{i-1}'', for $i=2,3,\dots,r-1$. By Proposition 3.9 we have that these numbers are given by the rank of the matrices $\boldsymbol{M}_i$.

We now introduce the polynomial ring $S=k[\eta_m^{(1)},\eta_m^{(2)},\dots,\eta_m^{(s)}]_{m\in\mathcal{M}_c}$, where $\eta_m^{(j)}$ are independent indeterminates over k.

For each $i=2,3,\dots,r-1$ there is a greatest integer N_i such that there is some non-zero $N_i\times N_i$-minor of $\boldsymbol{M}_i$. This means that there is a non-empty open set U' in the affine space with coordinate ring S such that $\operatorname{rank} \boldsymbol{M}_i(\lambda_m^{(j)}) = N_i$ for $(\lambda_m^{(j)})$ in U'. Hence there is a non-empty open set U in the Grassmannian parametrizing codimension s subspaces of R_c such that $V\in U$ implies that the Betti numbers of $A=R/\bigoplus_{d=0}^{\infty}(V:R)_d$, are given by $b_i' = \binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i} - N_i$. □

3.2. *Conjecture on the generic Betti numbers of compressed algebras*

After having established the existence of generic Betti numbers of compressed level algebras, we would like to determine these numbers. We conjecture that the generic Betti numbers are as small as we can hope for, that is that there are no relations between the rows and columns of the matrices described in the previous section except for the ones given by the Koszul complex.

In order to prepare a guess for the generic Betti numbers of compressed level algebras, we will study the numbers that occur in Proposition 3.3. Therefore we introduce the following notation.

Notation 3.11. For any positive integers r, s and c we define the sequence of numbers $d_1, d_2, \ldots, d_r$ by

$$\begin{aligned} d_i = &\binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i} \\ &- s\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1} \quad \text{for } i = 1,2,\ldots,r, \end{aligned} \tag{3.22}$$

where t is the least positive integer such that $\binom{t+r-1}{r-1} > s\binom{c-t+r-1}{r-1}$.

By Proposition 3.3 we have that $b_i' - b_{i-1}'' = d_i$, for $i = 1,2,\ldots,r$. Since $b_i' \geq 0$ and $b_i'' \geq 0$ for all i, we have that they are simultaneously minimized if one of b_i' and b_{i-1}'' is zero for each i. That means that we have $b_i' = \max\{0, d_i\}$ and $b_{i-1}'' = \max\{0, -d_i\}$ for all $i = 1,2,\ldots,r$.

It is also possible to see the problem of finding the generic Betti numbers as a problem of finding the maximal possible rank of the matrices $\boldsymbol{M}_i$ defined in the previous section. Since the dimension of $\operatorname{im}(\delta_i)$ is $\binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i}$ by Proposition 2.4 we have that $\operatorname{rank} \boldsymbol{M}_i \leq \binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i}$. By symmetry we have the same kind of relations between the rows of $\boldsymbol{M}_i$ as between the columns, and we therefore have that $\operatorname{rank} \boldsymbol{M}_i \leq s\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1}$. These two inequalities for the rank of $\boldsymbol{M}_i$ show that the dimension of the null space of $\boldsymbol{M}_i$ is at least $\max\{0, d_i\}$.

By Proposition 2.2(vi) we have that $n_{1,i} < n_{1,i+1}$. This means that $b_i' = 0$ implies that $b_{i+1}' = 0$. Hence, if $\max\{0, d_i\}$ is to give plausible candidates for the Betti numbers of A, we must have that $d_{i+1} \leq 0$ whenever $d_i \leq 0$. This is the case as we can see in the following proposition.

Proposition 3.12. *Given positive integers r, s and c, there is an integer i_0 such that $d_i > 0$ for $i < i_0$ and $d_i < 0$ for $i > i_0$.*

Proof. If $t = c+1$, the second term of d_i vanishes for $i = 1,2,\ldots,r-1$. Hence we can assume that $t \leq c$.

We see that $d_1 = \binom{t+r-1}{r-1} - s\binom{c-t+r-1}{r-1}$, and $d_r = \binom{t-1+r-1}{r-1} - s\binom{c-t+r}{r-1}$. Thus the definition of t yields that $d_1 > 0$ and $d_r \leq 0$. Now we write the expression for d_i in a new fashion.

$$\begin{aligned} d_i &= \binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i} - s\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1} \\ &= \frac{(t-1+i-1)!(t-1+r)!}{(t-1)!(i-1)!(t-1+i)!(r-i)!} \\ &\quad - s\frac{(c-t+r-i)!(c-t+r)!}{(r-i)!(c-t)!(i-1)!(c+1-t+r-i)!} \end{aligned}$$

$$= \frac{t(t-1+r)!}{(t-1+i)t!(r-1)!}\binom{r-1}{i-1} - s\frac{(c+1-t)(c-t+r)!}{(c+1-t+r)(c+1-t)!(r-1)!}\binom{r-1}{i-1}$$

$$= \binom{r-1}{i-1}\left[\binom{t-1+r}{r-1}\frac{t}{t+i-1} - s\binom{c-t+r}{r-1}\frac{c+1-t}{c+1-t+r-i}\right]. \qquad (3.23)$$

It is now easy to see that the expression inside the brackets at the last step of (3.23) is decreasing in i, since it is a difference between a decreasing function and an increasing function. The assertion of the proposition follows because the factor outside the bracket is positive. □

Conjecture 3.13. Let k be a field of characteristic 0. Assume that $r > 3$ and that $c > 1$. Let V be a subspace of R_c of codimension s in general position and let $A = R/I$ where $I = \bigoplus_{d=0}^{\infty}(V : R)_d$. Then the Betti numbers, $b_i = b_i' + b_i''$, of A are given by

$$b_i' = \max\left\{0, \binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i} - s\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1}\right\},$$

$$b_i'' = \max\left\{0, s\binom{c-t+r-i-1}{r-i-1}\binom{c-t+r}{i} - \binom{t-1+i}{i}\binom{t-1+r}{r-i-1}\right\}, \qquad (3.24)$$

for $i = 1, 2, \ldots, r-1$, where t is the initial degree of I.

Remark 3.14. In Sections 3.3 and 3.4 we will show that Conjecture 3.13 can be proved in some cases and in Section 4 we will present computer generated evidence for the conjecture. In fact, we would not have stated the conjecture if it had not been for these examples.

We can also formulate the conjecture in the following way: there exists an integer i such that $b_{i+1}' = b_{i-1}'' = 0$. If such an integer exists, we have by Proposition 2.2(viii) that the Betti numbers are determined by the Hilbert series. Hence the Betti numbers must be the ones given in Conjecture 3.13. Moreover, by Proposition 3.12, the converse is also true.

We will now give a couple of examples to show that Conjecture 3.13 does not hold if the assumptions on the embedding dimension and on the characteristic are dropped.

Example 3.15 (*cf. Schreyer* [19]). Let $r = 5$, $s = 1$ and $c = 2t - 1$. Thus we are in the Gorenstein codimension 5 case with socle in odd degree. The rank of the matrix $\boldsymbol{M}_3$ is, according to Conjecture 3.13, supposed to be $\binom{t-1+2}{2}\binom{t-1+5}{2} = \frac{t(t+1)(t+3)(t+4)}{4}$, which is odd if $t \equiv 2 \pmod 4$. However, in characteristic 2 we have that $\boldsymbol{M}_3$ is anti-symmetric – in a suitable ordering of rows and columns. Hence the rank cannot be odd and $\boldsymbol{M}_3$ drops rank by at least 1.

Example 3.16. Let $r=3$, $s=1$ and $c=2t-1$. By a result by Buchsbaum and Eisenbud [7, Corollary 2.2] the minimal number of generators of a Gorenstein ideal of codimension 3 has to be odd. According to Conjecture 3.13, the number of generators would be $\binom{t-1+0}{0}\binom{t-1+3}{2}-\binom{t-1+2}{2}\binom{t-1+3}{0}=t+1$, which is impossible if t is odd. Hence there has to be at least one generator in degree $t+1$ if t is odd.

3.3. Connections to generic forms

In some cases we have that the generic Betti numbers are obtained by algebras which are quotients of R by generic forms. In these cases we can use a result of Hochster and Laksov [15, Theorem 1] to verify Conjecture 3.13. Here we state a slightly weaker result than the one proved in [15].

Proposition 3.17. *If V is a subspace in general position in R_c with $r\dim_k V\leq \dim_k R_{c+1}$. Then there are no linear syzygies among the generators of V.*

Lemma 3.18. *Let V be a codimension s subspace in general position in R_c. If $rs\geq \dim_k R_{c-1}$ then there are no generators of $\bigoplus_{d\geq 0}(V:R)_d$ in degree less than c.*

Proof. From Proposition 1.4 follows that

$$\operatorname{codim}_k((V:R)_{c-1},R_{c-1})=\min\{\dim_k R_{c-1},rs\}. \tag{3.25}$$

Hence $\operatorname{codim}_k((V:R)_{c-1},R_{c-1})=\dim_k R_{c-1}$ and $(V:R)_{c-1}=0$, which proves the lemma. □

Proposition 3.19. *Proposition* 3.17 *is equivalent to Conjecture* 3.13 *in the case where* $r\dim_k V\leq \dim_k R_{c+1}$.

Proof. Assume that $r\geq 2$. Let V be a subspace of R_c such that $r\dim_k V\leq \dim_k R_{c+1}$ and let $I=\bigoplus_{d\geq 0}(V:R)_d$. Then we have that

$$\begin{aligned}rs&=r(\dim_k R_c-\dim_k V)\geq r\dim_k R_c-\dim_k R_{c+1}\\&=\left(r-\frac{(r-1+c+1)}{c+1}\right)\frac{r-1+c}{c}\dim_k R_{c-1}\\&=\frac{(r-1)(r-1+c)}{c+1}\dim_k R_{c-1}\geq \dim_k R_{c-1},\end{aligned} \tag{3.26}$$

since $r\geq 2$. Hence we can apply Lemma 3.18 to see that the initial degree of I is c.

From Proposition 3.17 we get that $b_2'=0$ since there are no linear syzygies among the forms of degree c in I. Thus the Betti numbers of $A=R/I$ have to be minimal given the Hilbert function and therefore Conjecture 3.13 holds in this case.

On the other hand if the conjecture is true, we get that

$$b_2' = \max\{0, d_2\} = \max\left\{0, c\binom{c-1+r}{r-2} - sr\right\}, \tag{3.27}$$

which is zero since $c\binom{c+r-1}{r-2} = r\dim_k R_c - \dim_k R_{c+1} = r(s + \dim_k V) - \dim_k R_{c+1} \le rs$. Hence Proposition 3.17 is a consequence of Conjecture 3.13 in this case. □

3.4. Connections to generic points in projective space

There is a conjecture similar to Conjecture 3.13 for a generic set of points in projective space. This conjecture was stated by Lorenzini [18] and is usually called the *Minimal Resolution Conjecture*, or MRC for short. Another conjecture concerning ideals of generic points is the *Ideal Generation Conjecture*, IGC, formulated by Geramita and Orecchia [13]. The MRC implies the IGC. There are cases where the MRC seems to be false, e.g. 11 points in general position in $\boldsymbol{P}^6$, where no calculation so far has given the conjectured Betti numbers. However, it is not yet proved that this is really a counterexample. To do this one has to prove that the open set where the Betti numbers are the expected ones is empty, which cannot be verified with only a few examples where random points fail to have these Betti numbers. There are no known counterexamples to the IGC.

Setup 3.20. Let X be a set of n points in $\boldsymbol{P}^r$. Then the homogeneous coordinate ring A of X has dimension 1 and its Hilbert function is bounded above by $\max\{n, \dim_k S_d\}$, where $S = k[x_0, x_1, \dots, x_r]$ is the homogeneous coordinate ring of $\boldsymbol{P}^r$. If X is in generic position the Hilbert function of A is equal to $\max\{n, \dim_k S_d\}$. We can assume that no points of X lie on the hypersurface $x_0 = 0$, so that x_0 is not a zero-divisor in A. Let $R = k[x_1, x_2, \dots, x_r] = S/(x_0)$. Let c be the least integer such that $\binom{r+c}{r} \ge n$ and let $s = n - \binom{r+c-1}{r}$. We then have that $s \le \binom{r+c}{r} - \binom{r+c-1}{r} = \binom{r-1+c}{r-1}$.

It is easy to see that the Hilbert function of the artinian reduction $B = A/(x_0)$ in the generic case is equal to the Hilbert function of a compressed Artin level algebra. In fact we have that $H_B(d) = H_R(d)$, for $d < c$, $H_B(c) = s$ and $H_B(d) = 0$, for $d > c$. If B is a level algebra, we have the inequality $H_B(d) \le \min\{H_R(d), sH_R(c-d)\}$. In particular we have that $H_B(c-1) = H_R(c-1) \le sH_R(1) = sr$. It follows from the following theorem due to Trung and Valla [21] on the Cohen–Macaulay type of the coordinate ring of a set of points in generic position that it is also true that B is level if this inequality holds.

Theorem 3.21 (Trung–Valla [21]). *Let B be the Artinian reduction of the coordinate ring A of a set of n generic points in $\boldsymbol{P}^r$. Then*

$$\dim_k \operatorname{Soc}(B)_{c-1} = \max\left\{0, \binom{r+c-2}{r-1} + r\binom{r+c-1}{r} - rn\right\}, \tag{3.28}$$

where c is the highest degree such that $B_c \ne 0$.

Corollary 3.22. *If $H_R(c-1) \leq rs$ the coordinate ring of a set of n generic points in $\boldsymbol{P}^r$ is a compressed level algebra with socle of dimension s in degree c.*

Proof. Since we have that $s = n - \binom{r+c-1}{r}$ we have from the theorem that

$$\dim_k \mathrm{Soc}(B) = \max\{0, H_R(c-1) - sr\}. \tag{3.29}$$

Hence we get that $\mathrm{Soc}(B)_{c-1} = 0$ if the assumption of the corollary is satisfied. Since $H_B(c-1) = H_R(c-1)$, we have that the socle is a priori located in degrees $c-1$ and c, it follows that B is a compressed level algebra. □

Conjecture 3.23 (*Minimal Resolution Conjecture, Lorenzini* [18]). Let A be the homogeneous coordinate ring of a set of n points in generic position in $\boldsymbol{P}^r$. Then there exists an integer i such that $\mathrm{Tor}^S_{i+1}(A,k)_{i+c} = 0$ and $\mathrm{Tor}^S_{i-1}(A,k)_{i+c-1} = 0$, where c is the least degree for which $n \leq \dim_k S_c$.

Thus the MRC gives the same Betti numbers as Conjecture 3.13 in the case described in Corollary 3.22. It it follows that under these circumstances the MRC implies Conjecture 3.13.

Since the MRC is proved for some values of r and n we have that Conjecture 3.13 is also proved if n satisfies the condition in Corollary 3.22.

The Minimal Resolution Conjecture has been proved in the following cases:

- For any number of points in $\boldsymbol{P}^2$ by Geramita and Maroscia [12].
- For any number of points in $\boldsymbol{P}^3$ by Ballico and Geramita [1].
- For $\binom{r+2}{2} - r$ points in $\boldsymbol{P}^r$ by Lorenzini [18].
- For $r+1$, $r+2$, $r+3$ and $r+4$ points in $\boldsymbol{P}^r$ by Lorenzini [18], by Geramita and Lorenzini [11] and Cavaliere et al. [8].
- For $r \leq 9$ and $n \leq 50$, except for the cases $(r,n) = (6,11)$, $(7,12)$ and $(8,13)$, by Beck and Kreuzer [3].
- For large numbers of points in any $\boldsymbol{P}^r$ by Hirschowitz and Simpson [14].

3.5. Generic Betti numbers of Gorenstein Artin algebras in embedding dimension 4

We now prove that Conjecture 3.13 holds for Gorenstein Artin algebras of embedding dimension 4. For this we will use that the Minimal Resolution Conjecture has been proved in $\boldsymbol{P}^3$ and that the canonical module of the coordinate ring of points can be identified with an ideal of the ring itself. Assume that the characteristic of k is 0.

Proposition 3.24. *The generic Betti numbers of a Gorenstein Artin algebra of embedding dimension 4 with initial degree t and socle degree $2t-1$ are*

$$\begin{pmatrix} \binom{t+2}{2} & \binom{t+2}{2} - 1 & 0 \\ 0 & \binom{t+2}{2} - 1 & \binom{t+2}{2} \end{pmatrix} \tag{3.30}$$

Proof. We will use the result of Ballico and Geramita [1] which shows that the Minimal Resolution Conjecture holds for points in $\boldsymbol{P}^3$. We first show that there is a set of points in generic position in $\boldsymbol{P}^3$ whose Betti numbers are given by

$$\begin{pmatrix} b_1' & b_2' & 0 \\ 0 & b_2'' & b_3'' \end{pmatrix}. \tag{3.31}$$

Let X be a set of $\binom{t-1+r}{r}+s$ points in generic position in $\boldsymbol{P}^3$. Let $R=k[x_0,x_1,x_2,x_3]$ be the homogeneous coordinate ring of $\boldsymbol{P}^3$ and let $A(X)=R/I(X)$ be the coordinate ring of X. Then the initial degree of the ideal $I(X)\subseteq R$ is t. Since the Minimal Resolution Conjecture holds in $\boldsymbol{P}^3$, or in fact by Theorem 3.21, we have that $\operatorname{Tor}_3^R(A(X),k)$ is concentrated in degree $t+3$, i.e. $b_3'=0$, if

$$\binom{t+1}{2} - 3s \le 0. \tag{3.32}$$

Furthermore, by the same result, $\operatorname{Tor}_1^R(A(X),k)$ is concentrated in degree t, i.e. $b_1''=0$, if there are sufficiently many generators in degree t to span the ideal in degree $t+1$. That is, if

$$3\left[\binom{2+t}{2} - s\right] \ge \binom{t+3}{2}. \tag{3.33}$$

The inequalities (3.32) and (3.33) together show that the Betti numbers of $A(X)$ have the form (3.31) if

$$\frac{(t+1)t}{6} \le s \le \frac{t(t+2)}{3}. \tag{3.34}$$

There has to be an integer s in this range if $t\ge 2$, since the length of the interval is $t(t+3)/6$.

Now we use the fact that the canonical module ω_X of $A(X)$ can be embedded as an ideal $\omega_X\subseteq A(X)$ of initial degree t (cf. [4]). This means that we have a short exact sequence

$$0 \to \omega_X \to A(X) \to A \to 0, \tag{3.35}$$

and the corresponding long exact sequence of Tor^R splits up in the following way

$$\begin{array}{l} 0 \to \operatorname{Tor}_4^R(A,k) \to \operatorname{Tor}_3^R(\omega_X,k) \to 0 \\ 0 \to \operatorname{Tor}_3^R(A(X),k) \to \operatorname{Tor}_3^R(A,k) \to \operatorname{Tor}_2^R(\omega_X,k) \to 0 \\ 0 \to \operatorname{Tor}_2^R(A(X),k) \to \operatorname{Tor}_2^R(A,k) \to \operatorname{Tor}_1^R(\omega_X,k) \to 0, \\ 0 \to \operatorname{Tor}_1^R(A(X),k) \to \operatorname{Tor}_1^R(A,k) \to \operatorname{Tor}_0^R(\omega_X,k) \to 0 \\ 0 \to \operatorname{Tor}_0^R(A(X),k) \to \operatorname{Tor}_0^R(A,k) \to 0 \end{array} \tag{3.36}$$

since $\operatorname{Tor}_3^R(\omega_X,k)$ lies in degree $2t+3$, $\operatorname{Tor}_3^R(A(X),k)$ and $\operatorname{Tor}_2^R(\omega_X,k)$ lie in degree $t+3$, $\operatorname{Tor}_2^R(A(X),k)$ and $\operatorname{Tor}_1^R(\omega_X,k)$ lie in degrees $t+1$ and $t+2$, $\operatorname{Tor}_1^R(A(X),k)$ and $\operatorname{Tor}_0^R(\omega_x)$ lie in degree t, and $\operatorname{Tor}_0^R(A(X),k)$ lies in degree 0. In this way we can compute the Betti numbers of A in terms of the Betti numbers of $A(X)$. In particular, we see that $\operatorname{Tor}_1^R(A,k)$ is concentrated in degree t, $\operatorname{Tor}_3^R(A,k)$ is concentrated in degree $t+3$. It follows from Proposition 2.6 that A is a Gorenstein Artin algebra with socle in degree $2t-1$, since $\operatorname{Tor}_4^R(A,k)=k(-2t-3)$. Hence by Proposition 3.2, we conclude that A is compressed. The generic Betti numbers are now given by Proposition 3.3. □

4. Computational evidence for Conjecture 3.13

The main reason for stating Conjecture 3.13 is that it holds for a great number of examples. Most of those are obtained by computer experiments by means of the computer algebra system Macaulay [2]. Since the conjecture concerns the generic Betti numbers of compressed algebras, it suffices to find one algebra for each set of parameters (r,c,s) whose Betti numbers agree with the conjectured Betti numbers. We can by some observations about the generic Betti numbers of level algebras reduce the number of cases to consider dramatically. More precisely, we only have to check the conjecture for a few values of the socle dimension s for each choice of embedding dimension r and socle degree c.

In this section we will give a list of the cases where the Conjecture 3.13 has been verified by computer calculations.

4.1. Theoretical tools for the computations

Since the computations of resolutions by Gröbner bases are rather time consuming, it is valuable to reduce the number of such calculations to a minimum.

First we observe that for fixed r and c, we can find pairs of integers s_1 and s_2 such that if the conjecture holds for (r,c,s_1) and for (r,c,s_2) then it holds for all (r,c,s) with s between s_1 and s_2.

Moreover, we do not need to compute all the Betti numbers to verify the conjecture, but only the linear part of the resolution, or the numbers b_i', for $i=1,2,\ldots,r-1$.

Lemma 4.1. *Let $A=R/I$ and $A'=R/I'$ be compressed level algebras with parameters (r,c,s) and (r,c,s') respectively and whose Betti numbers are the generic Betti numbers. Suppose that the initial degree of I and I' is t and that $s<s'$. Then we have that*

$$\dim_k \operatorname{Tor}_i^R(A,k)_{i+t-1} \geq \dim_k \operatorname{Tor}_i^R(A',k)_{i+t-1}, \tag{4.1}$$

$$\dim_k \operatorname{Tor}_i^R(A,k)_{i+t} \leq \dim_k \operatorname{Tor}_i^R(A',k)_{i+t} \tag{4.2}$$

for $i=1,2,\ldots,r$.

Proof. We recall from Definition 3.7 the matrices $\boldsymbol{M}_i$. Let $\boldsymbol{M}_i$ and $\boldsymbol{M}_i'$ be the matrices corresponding to the integers s and s', respectively. Then we have that $\boldsymbol{M}_i$ is a submatrix of $\boldsymbol{M}_i'$, for all $i=2,3,\dots,r$. By Proposition 2.4 we can delete rows and columns of $\boldsymbol{M}_i$ and $\boldsymbol{M}_i'$ so that their sizes are $s\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1}\times\binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i}$ and $s'\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1}\times\binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i}$.

By Proposition 3.9 we have that

$$\dim_k \operatorname{Tor}_i^R(A,k)_{i+t-1} - \dim_k \operatorname{Tor}_i^R(A',k)_{i+t-1} = \operatorname{rank}\boldsymbol{M}_i' - \operatorname{rank}\boldsymbol{M}_i \geq 0, \tag{4.3}$$

where the inequality holds since $\boldsymbol{M}_i$ is a submatrix of $\boldsymbol{M}_i'$. Furthermore, we have that

$$\begin{aligned}&\dim_k \operatorname{Tor}_{i-1}^R(A',k)_{i+t-1} - \dim_k \operatorname{Tor}_{i-1}^R(A,k)_{i+t-1}\\ &\quad=(s'-s)\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1} + \operatorname{rank}\boldsymbol{M}_i - \operatorname{rank}\boldsymbol{M}_i'.\end{aligned} \tag{4.4}$$

When we have reduced the number of rows in $\boldsymbol{M}_i$ and $\boldsymbol{M}_i'$ as indicated above, we have that the number of rows added in $\boldsymbol{M}_i'$ relative to $\boldsymbol{M}_i$ is given by $(s'-s)\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1}$. Hence the rank cannot increase by more than this amount and the quantity in (4.4) is non-negative. □

As a consequence of Lemma 4.1, we have that we only need to verify Conjecture 3.13 for a few values of s for each choice of r and c. More precisely, if we have that $\operatorname{Tor}_i^R(A,k)_{i+t-1}=0$ for a generic level algebra with parameters (r,c,s), then $\operatorname{Tor}_i^R(A',k)_{i+t-1}=0$ for all generic level algebras A' with parameters (r,c,s') and $s'>s$. In the same way we have that if $\operatorname{Tor}^R(A,k)_{i+t}=0$, for a generic level algebra with parameters (r,s,c), then $\operatorname{Tor}_i^R(A',k)_{i+t}=0$ for all generic level algebras A' with parameters (r,c,s') and $s'<s$.

The application of these two implications is most easily illustrated by an example.

Example 4.2. Suppose that we have verified Conjecture 3.13 for the parameters $(r,c,s)=(4,4,6)$ and $(4,4,11)$. This means that we have found compressed level algebras A' and A'' with resolutions

$$0 \to R^6(-7) \to R^{14}(-6) \to R^{21}(-4) \to R^{14}(-3) \to R \to A' \to 0 \tag{4.5}$$

and

$$0 \to R^{11}(-7) \to R^{34}(-6) \to R^{30}(-5)\oplus R(-4) \to R^9(-3) \to R \to A'' \to 0. \tag{4.6}$$

By the notation of 3.4 the Betti numbers of these algebras are

$$\begin{pmatrix}14 & 21 & 0\\ 0 & 0 & 14\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}9 & 1 & 0\\ 0 & 30 & 34\end{pmatrix}. \tag{4.7}$$

Now let A be a compressed algebra with the generic Betti numbers and parameters $(r,c,s)=(4,4,s)$, where $6\leq s\leq 11$. Then we have from Lemma 4.1 that

$$\dim_k \operatorname{Tor}_3^R(A,k)_5 \leq \dim_k \operatorname{Tor}_3^R(A',k)_5 = 0, \tag{4.8}$$

since $s \geq 6$, and that

$$\dim_k \operatorname{Tor}_1^R(A,k)_4 \leq \dim_k \operatorname{Tor}_1^R(A'',k)_4 = 0, \tag{4.9}$$

since $s \leq 11$. Hence we have that the Betti numbers of A can be written as

$$\begin{pmatrix} b_1' & b_2' & 0 \\ 0 & b_2'' & b_3'' \end{pmatrix}. \tag{4.10}$$

The Betti numbers of A are, by Proposition 2.2(viii), determined by the Hilbert function of A and the resolution of A verifies Conjecture 3.13. Hence we do not need to actually compute the Betti numbers for $6 < s < 11$.

We will now give a summary of the cases where Conjecture 3.13 has been proved by means of computers. All these calculations has been done by Macaulay and hence modulo 31 991. The results are therefore valid in characteristic 31 991 and in characteristic 0.

4.1.1. Gorenstein Artin algebras

The case of Gorenstein Artin algebras is probably the most critical one for Conjecture 3.13. As we saw in Example 3.16 the conjecture is not even true for Gorenstein Artin algebras in embedding dimension three. For even socle degree, there is no problem since compressed Gorenstein Artin algebras of even socle degree are extremely compressed. Hence they have almost linear resolutions, by Proposition 3.6. We have computed the Betti numbers of generic compressed Gorenstein Artin algebras for several embedding dimensions and socle degrees and thereby verified Conjecture 3.13. The results of these computations are displayed in Table 1.

4.1.2. Artin level algebras of higher socle dimensions

For higher socle dimensions than 1 there are no counterexamples to Conjecture 3.13 in embedding dimension 3 and we have therefore included these cases in the Table 2. For the embedding dimensions and socle degrees in Table 2, we have verified the conjecture for all possible dimensions of the socle, using Lemma 4.1.

Table 1
Cases of compressed Gorenstein Artin algebras of odd socle degree where Conjecture 3.13 has been verified

Emb. dim.	Socle degree
5	3, 5, 7, 9, 11, 13
6	3, 5, 7, 9
7	3, 5, 7
8	3, 5
9	3
10	3

Table 2
Cases of compressed Artin level algebras where Conjecture 3.13 has been verified

Emb. dim.	Socle degree
3	$2 \leq c \leq 18$
4	$2 \leq c \leq 10$
5	$2 \leq c \leq 6$
6	$2 \leq c \leq 5$
7	$2 \leq c \leq 3$
8	$2 \leq c \leq 3$
9	$2 \leq c \leq 2$
10	$2 \leq c \leq 2$

References

[1] E. Ballico, A.V. Geramita, The minimal free resolution of s general points in $\boldsymbol{P}^3$, Can. Math. Soc. Conf. Proc. 6 (1986) 1–10.

[2] D. Bayer, M. Stillman, Macaulay: A system for computation in algebraic geometry and commutative algebra. Source and object code available for Unix and Macintosh computers. Contact the authors, or download from zariski.harvard.edu via anonymous ftp.

[3] S. Beck, M. Kreuzer, How to compute the canonical module of a set of points, in: algorithms in algebraic geometry and applications, Santander, 1994, Progress in Math., vol. 143, Birkhäuser, Basel, 1996, pp. 51–78.

[4] M. Boij, Gorenstein Artin algebras and points in projective space, Preprint, 1994.

[5] M. Boij, D. Laksov, Nonunimodality of graded Gorenstein Artin algebras, Proc. Amer. Math. Soc. 120 (1994) 1083–1092.

[6] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.

[7] D.A. Buchsbaum, D. Eisenbud, Algebra structure for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math. 99 (1977) 447–485.

[8] M.P. Cavaliere, M.E. Rossi, G. Valla, On the resolution of points in generic position, Comm. Algebra 19 (1991) 1083–1097.

[9] M.P. Cavaliere, M.E. Rossi, G. Valla, On the resolution of certain graded algebras, Trans. Amer. Math. Soc. 337 (1993) 389–409.

[10] R. Fröberg, D. Laksov, Compressed algebras, Lecture Notes in Math., vol. 1092, Springer, Berlin, 1984, pp. 121–151

[11] A.V. Geramita, A. Lorenzini, The Cohen–Macaulay type of $n+3$ points in $\boldsymbol{P}^n$, Queen's Papers in Pure and Appl. Math. vol. 83, 1989, pp. F1–F5.

[12] A.V. Geramtita, P. Maroscia, The ideal of forms vanishing at a finite set of points in $\boldsymbol{P}^n$, J. Algebra 90 (1984) 528–555.

[13] A.V. Geramita, F. Orecchia, Minimally generating ideals defining certain tangent cones, J. Algebra 78 (1982) 36–57.

[14] A. Hirschowitz, C. Simpson, Le résolution minimale de l'ideal d'un arrangement général d'un grand nombre de points dans $\boldsymbol{P}^n$, Preprint, 1994.

[15] M. Hochster, D. Laksov, The linear syzygies of generic forms, Comm. Algebra 15 (1987) 227–239.

[16] A. Iarrobino, Compressed algebras: Artin algebras having given socle degrees and maximal length, Trans. Amer. Math. Soc. 285 (1984) 337–378.

[17] A. Iarrobino, Associated graded algebra of a Gorenstein Artin algebra, Memoirs Amer. Math. Soc., vol. 514, American Mathematical Society, Providence, 1994.

[18] A. Lorenzini, The minimal resolution conjecture, J. Algebra 156 (1993) 5–35.

[19] F.O. Schreyer, Syzygies of canonical curves and special linear series, Math. Ann. 275 (1986) 105–137.

[20] R. Stanley, Cohen–Macaulay complexes, in: M. Aigner (Ed.), Higher Combinatorics, Reidel, Dordrecht, 1977, pp. 51–62.

[21] N.V. Trung, G. Valla, The Cohen–Macaulay type of points in generic position, J. Algebra 125 (1989) 110–119.